Moduli Spaces, Geodesics, and Flops

FABIAN RUEHLE (NORTHEASTERN UNIVERSITY & IAIFI)

String Pheno 2022
07/04/2022

Based on

[Brodie, Constantin, Lukas, FR: 2112.12106]

[Brodie, Constantin, Lukas, FR: 2108.10323]

[Brodie, Constantin, Lukas, FR: 2104.03325]

[Ashmore, FR: 2103.07472]

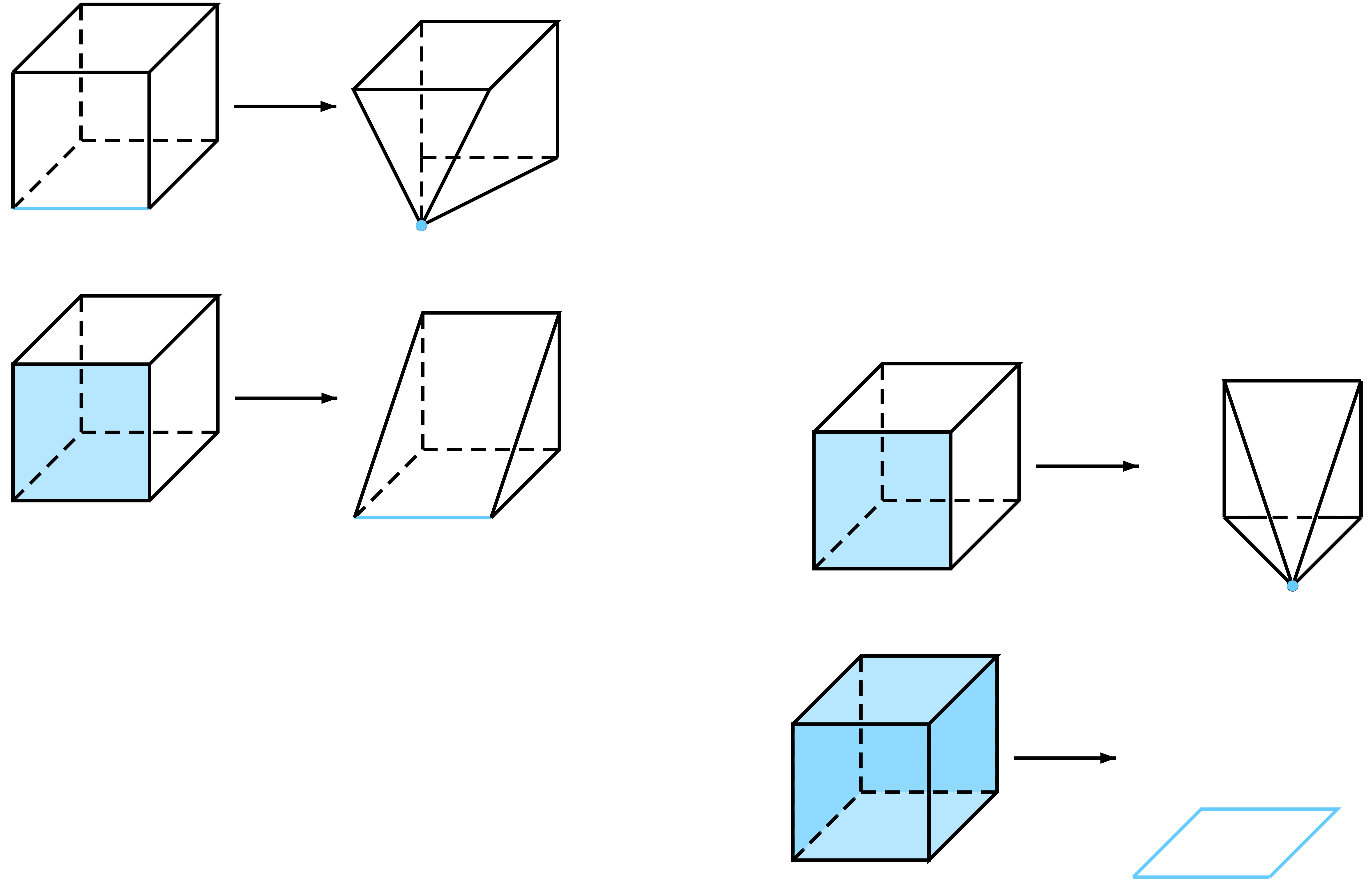
Motivation

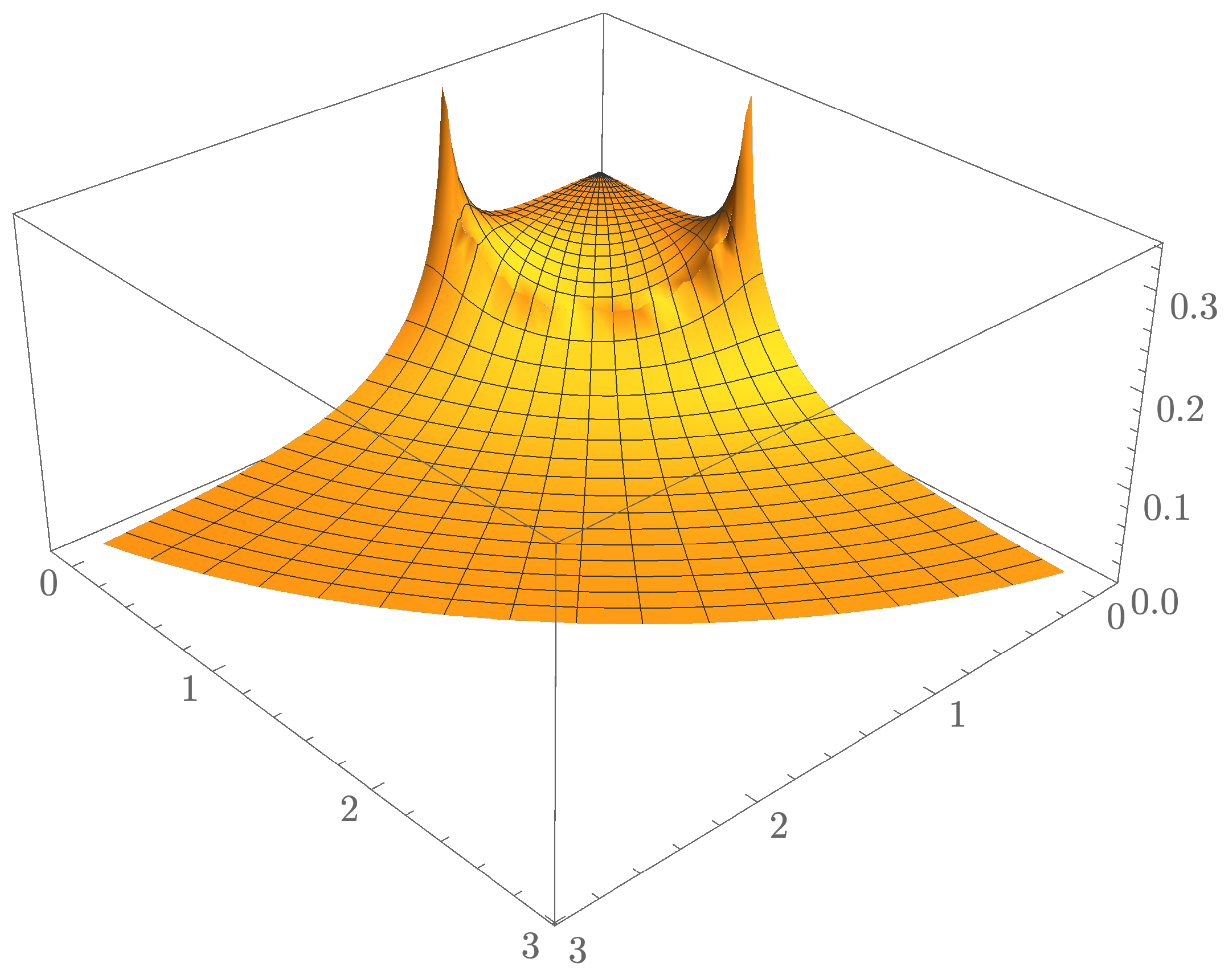
Swampland Distance Conjecture: [Ooguri, Vafa `06]

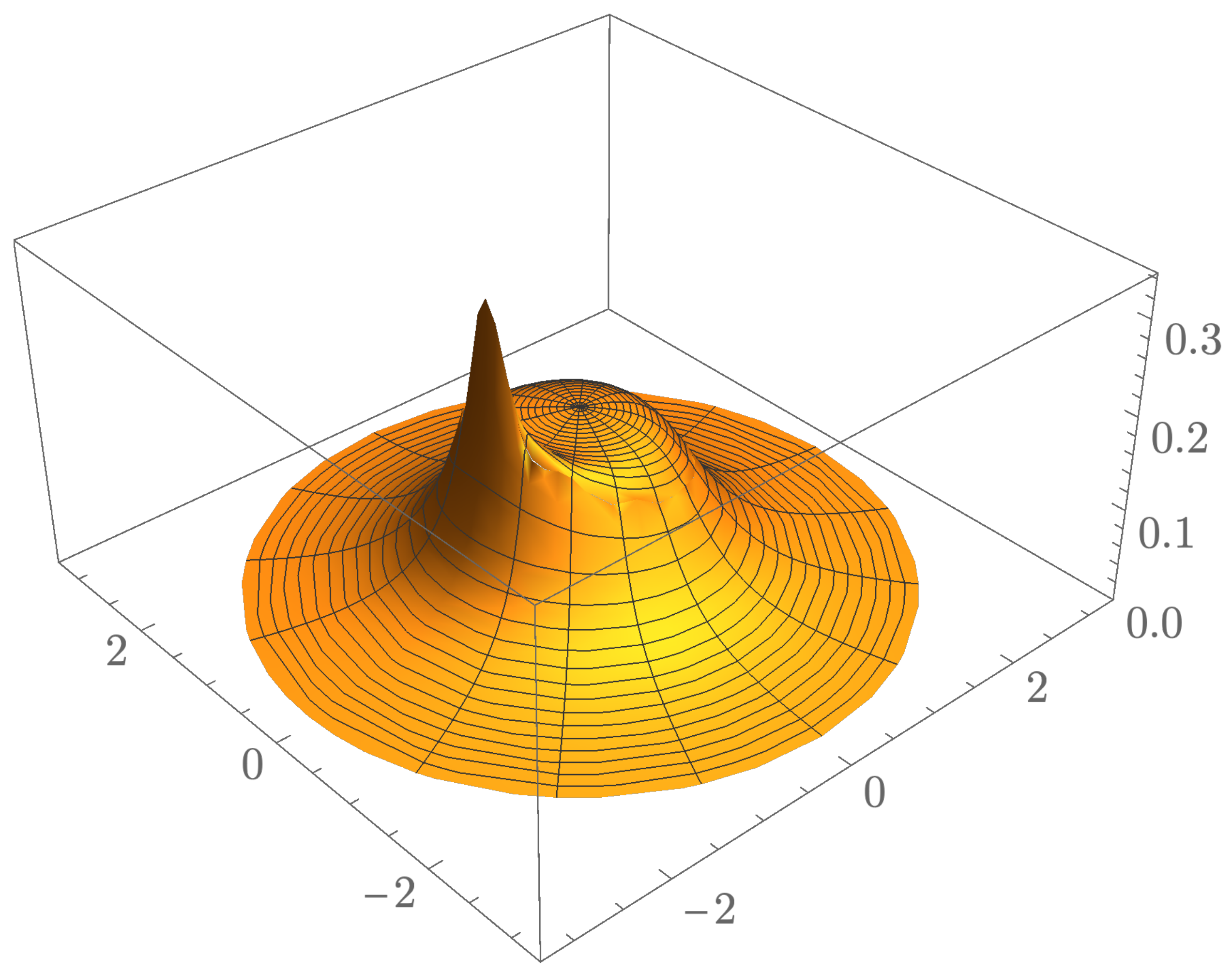
Compare a string theory compactified on a CY X at a point p_1 in its moduli space with the theory at a point p_0. Denote the (geodesic) moduli space distance by d.
Then, the theory at p_1 has an infinite number of light particles, with mass starting at the order $m \sim e^{-\alpha d}$ with $\alpha = \mathcal{O}(1)$ in Planck units.

Outline

- Structure of moduli spaces
- Massive towers and the Swampland Distance Conjecture
- The Kawamata-Morrison and the Swampland Distance Conjecture
- Conclusions

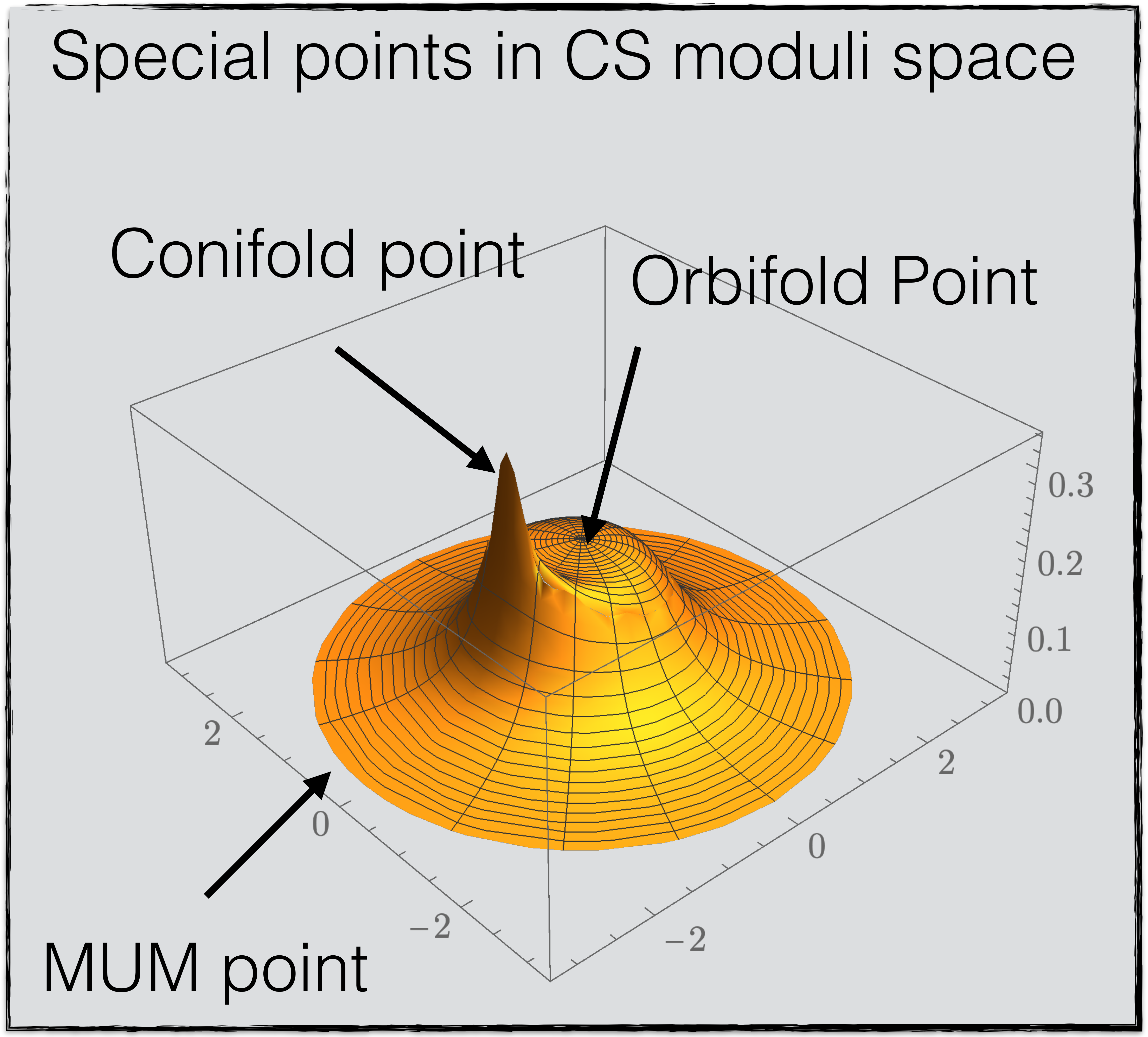






Structure of Moduli Spaces

The CS moduli space



- Metric for 1-parameter quintic

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4 = 0$$

- Degenerations classified via LMHS
[Grimm,Li,Palti `18; Grimm,Palti,Valenzuela `18; Blumenhagen,Kläwer, Schlechter `18; Grimm,FR,van de Heisteeg `19; Joshi,Klemm `19]

- Can be computed from the periods hypergeometric Picard-Fuchs system)
[Candelas,De La Ossa,Green,Parkes `91]

- Solve geodesic equation

$$\ddot{\gamma}(\tau) + \Gamma^c_{ab}\dot{\gamma}^a(\tau)\dot{\gamma}^b(\tau) = 0$$

- Compute geodesic distances

$$d(p_1, p_2) = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \sqrt{g_{a\bar{b}}(\gamma(\tau))\dot{\gamma}^a(\tau)\dot{\gamma}^b(\tau)}$$

The Kähler moduli space

- Kähler cone = cone of Kähler forms. All volumes are > 0

$$\mathrm{vol}(CY) = \int_X J^3 = \kappa_{ijk} t^i t^j t^k \,, \quad \mathrm{vol}(D_a) = \int_X J^2 D_a = \kappa_{ija} t^i t^j \,, \quad \mathrm{vol}(C_{ab}) = \int_X J D_a D_b = \kappa_{iab} t^i$$

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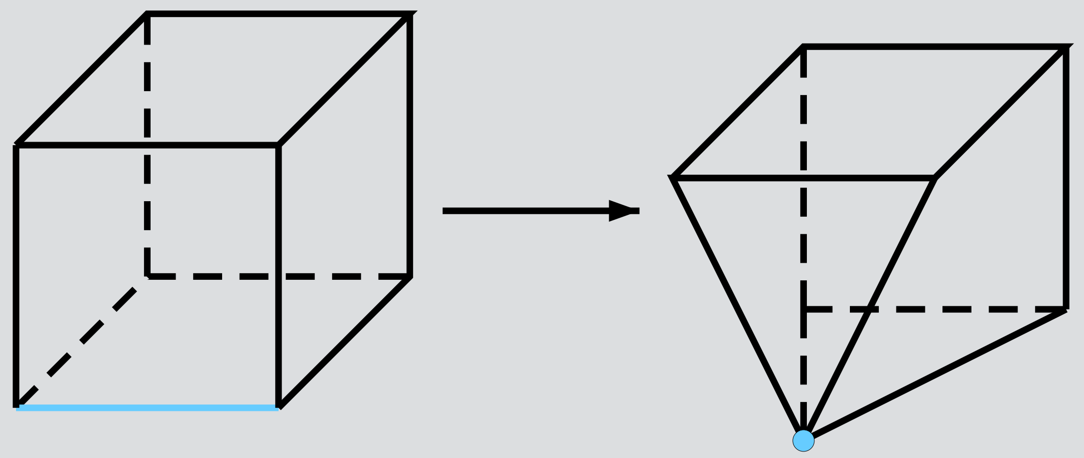
- At the wall of the KC, some volumes → 0: “primitive contractions”



Flop wall:
Curve collapses to a point

$$\text{vol}(C) \to 0$$

$$\text{vol}(D), \text{vol}(X) > 0$$

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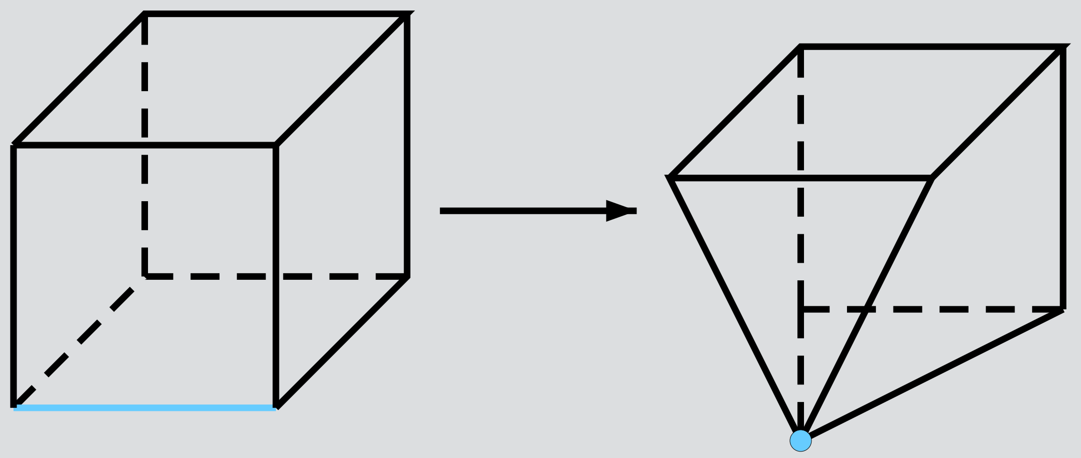

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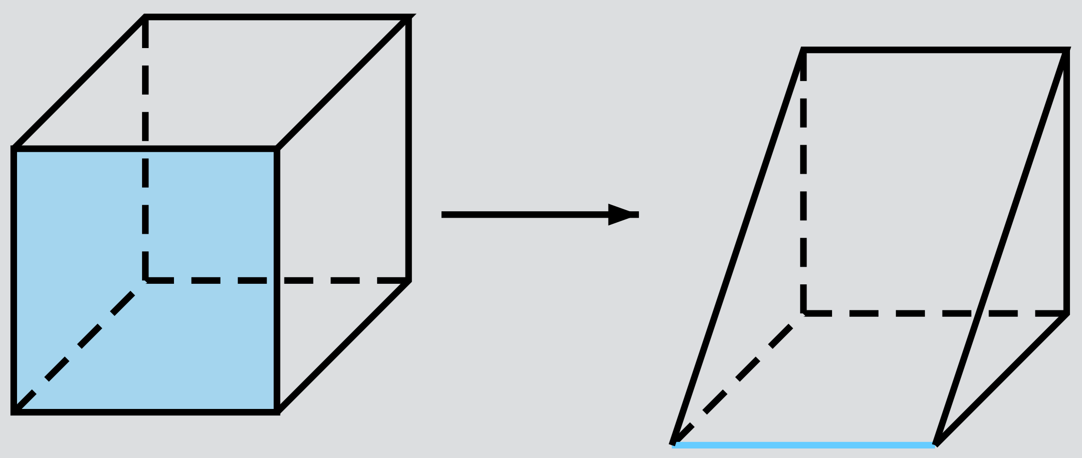

Zariski wall (a):
Divisor collapses to a curve, in general unclear how to go beyond

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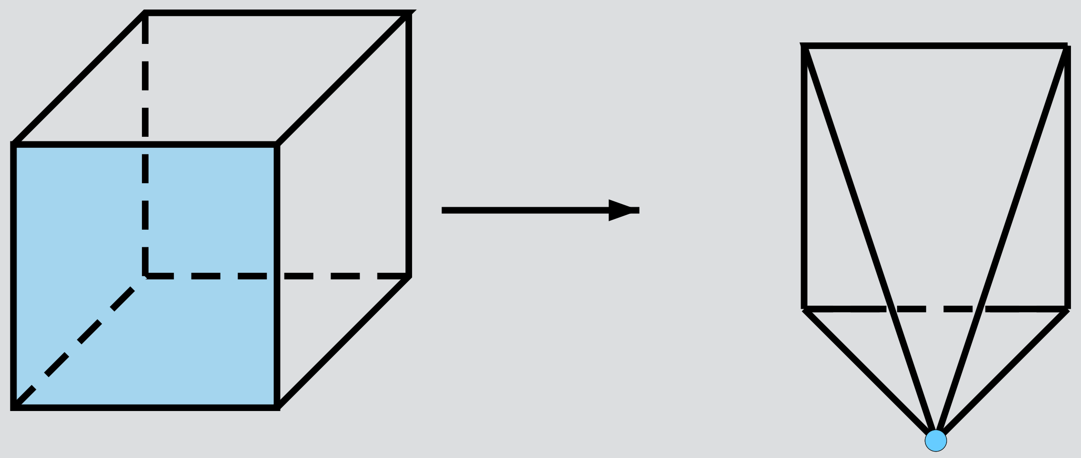

Zariski wall (b):
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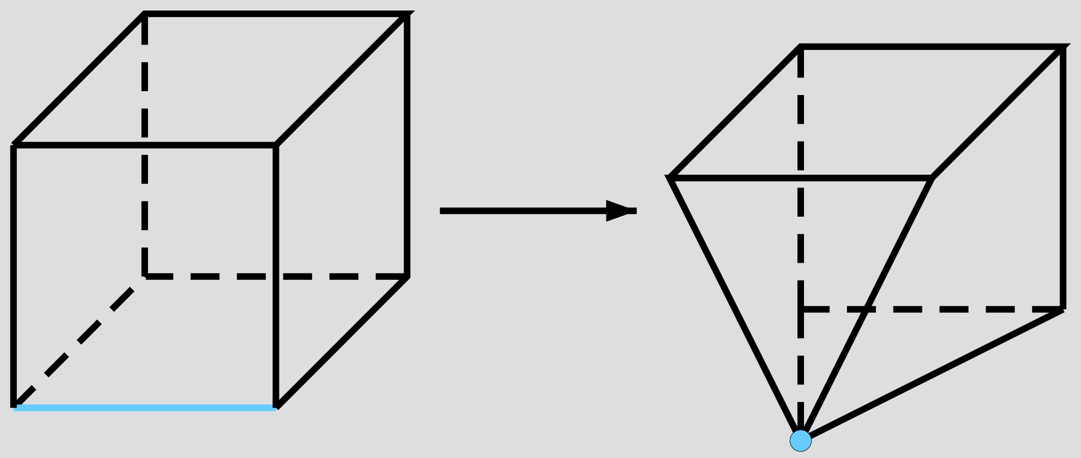

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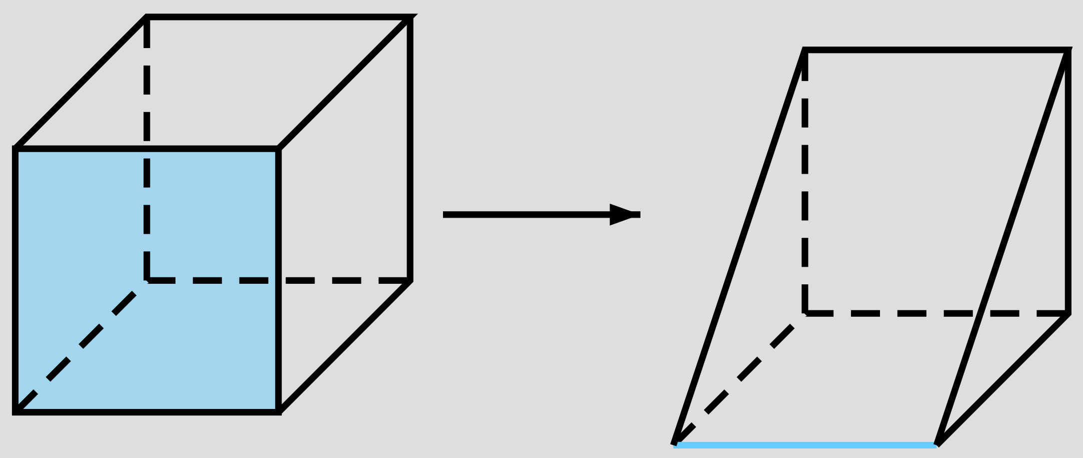

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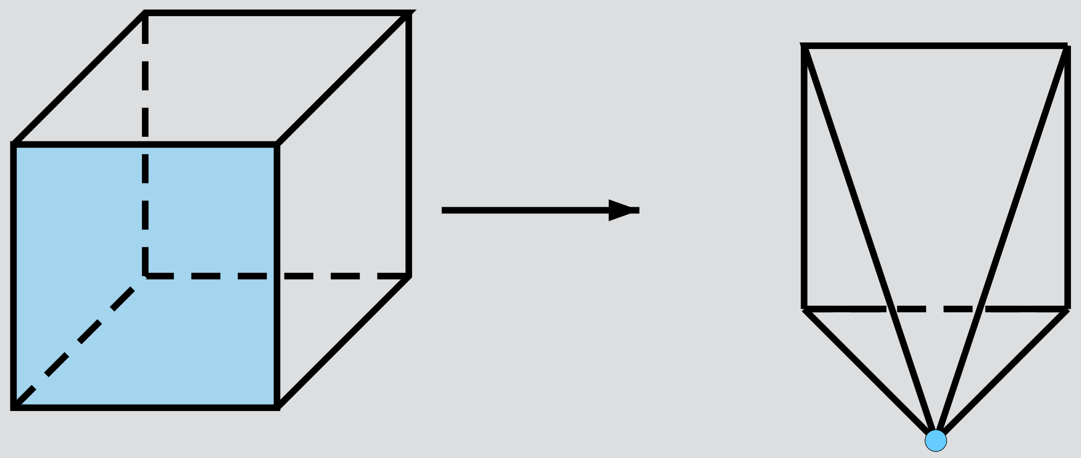

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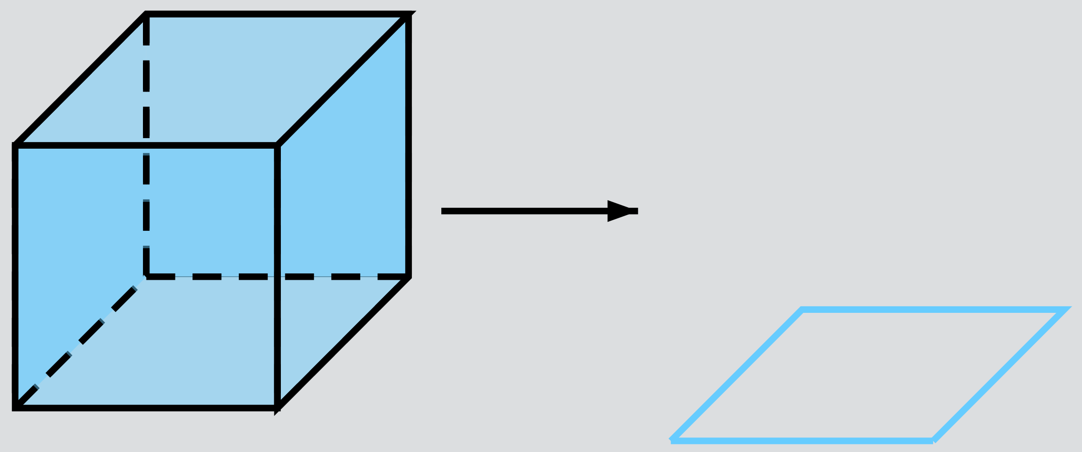
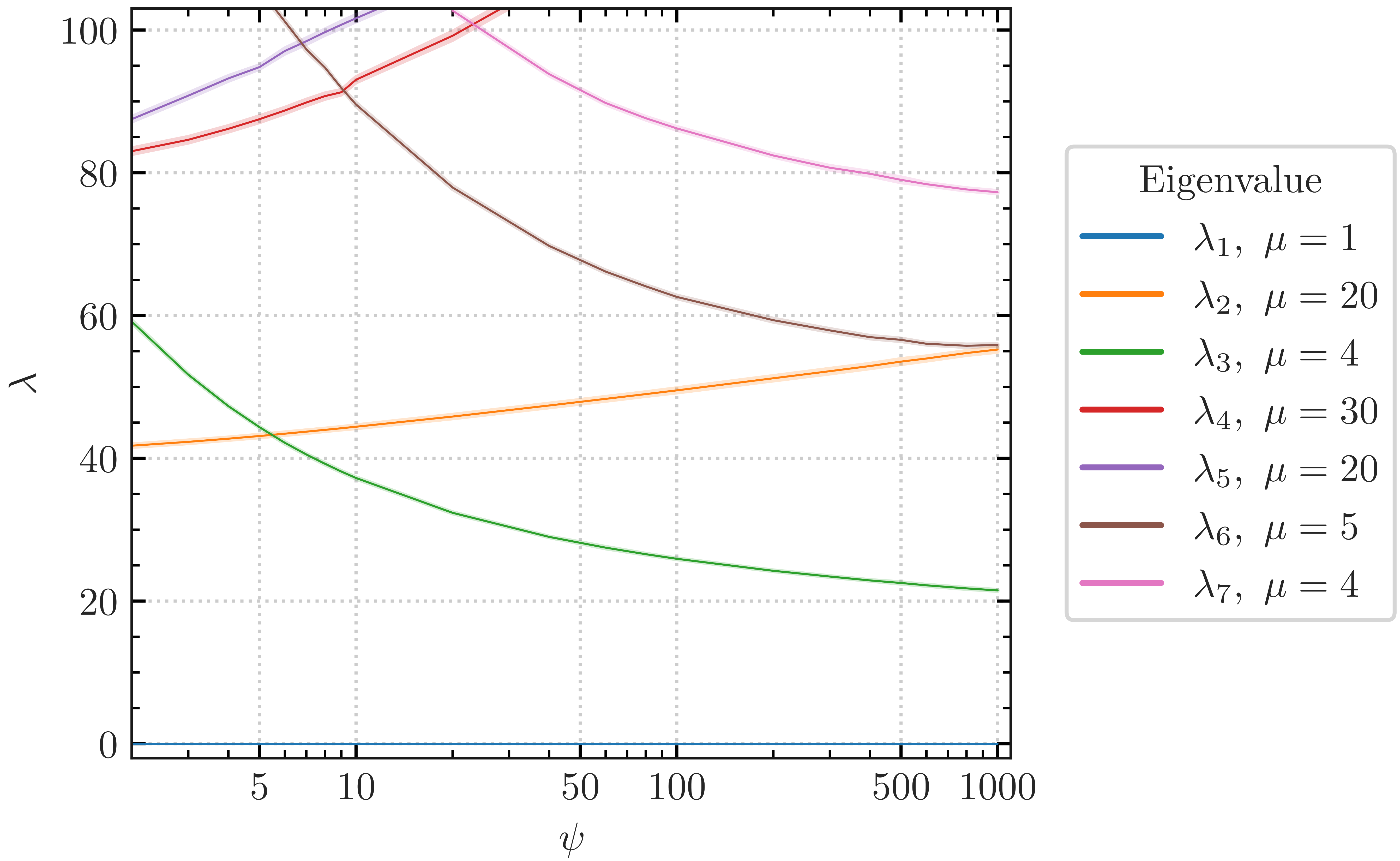
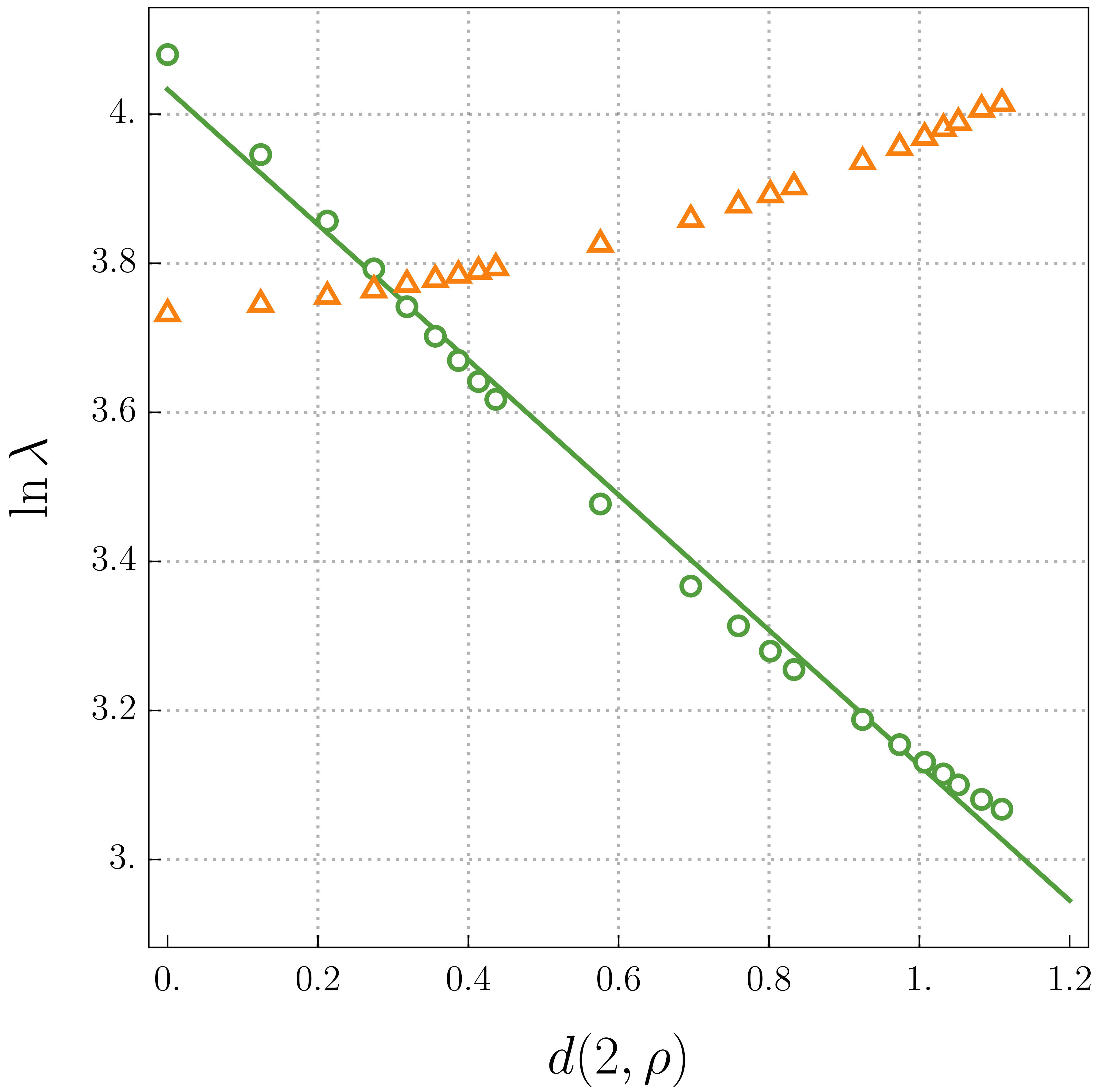


Effective cone wall:
CY collapses, KC ends

$\mathrm{vol}(X) \to 0$





Massive Towers and the SDC

Compute massive KK states (schematically)

- Starting point: **10D KG equation** $\Delta_{10D}\ \Phi_{10D} = 0 \qquad \Delta_{10D} \sim g^{MN}\partial_M\partial_N$
- Now decompose

$$\Delta_{10D} = (\Delta_{4D};\ \Box_{6D}) \qquad \Phi_{10D} = (\phi_{4D};\ \varphi_{6D}) \qquad g^{MN} = (g^{\mu\nu};\ g^{ab})$$

- Use **6D eigenfunctions** of d'Alembertian

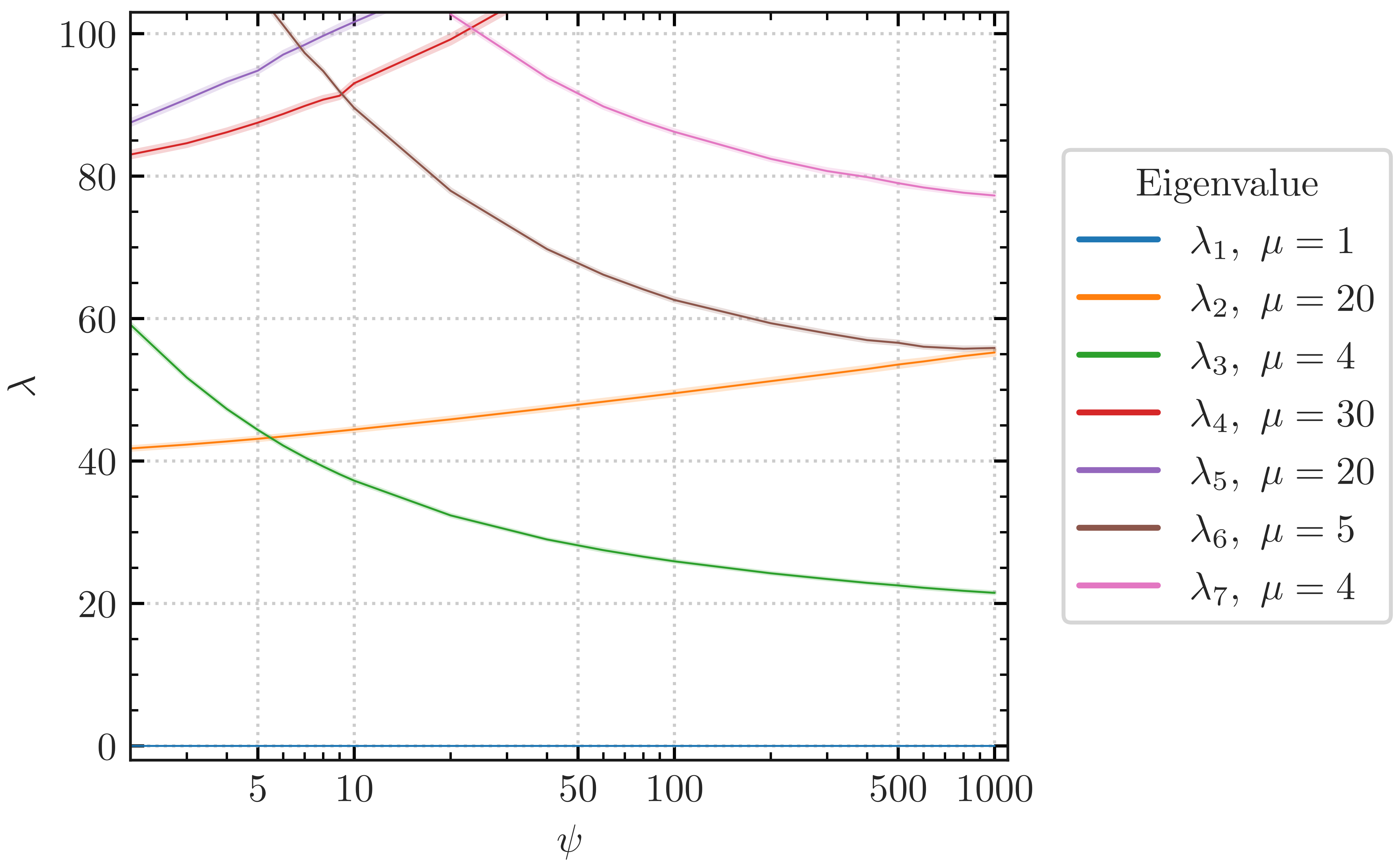
$$\Box_{6D}\varphi_{6D} = \lambda\varphi_{6D} \qquad \Box = d\delta + \delta d = \sqrt{\det(g)}^{-1}\partial_a\sqrt{\det(g)}g^{ab}\partial_b$$

- Get **4D KG equation** $(\Delta_{4D} + \lambda)\phi_{4D} = 0 \qquad m^2 \sim \lambda$
- Can compute CY metric using machine learning [see Andre's talk]

[Ashmore,He,Ovrut `19; Anderson,Gerdes,Gray,Krippendorf,Raghuram,FR `20; Douglas,Lakshminarasimhan,Qi `20; Jejjala,Mayorga Pena,Mishra `20; Larfors,Lukas,FR,Schneider `21 `22]

Massive KK states I

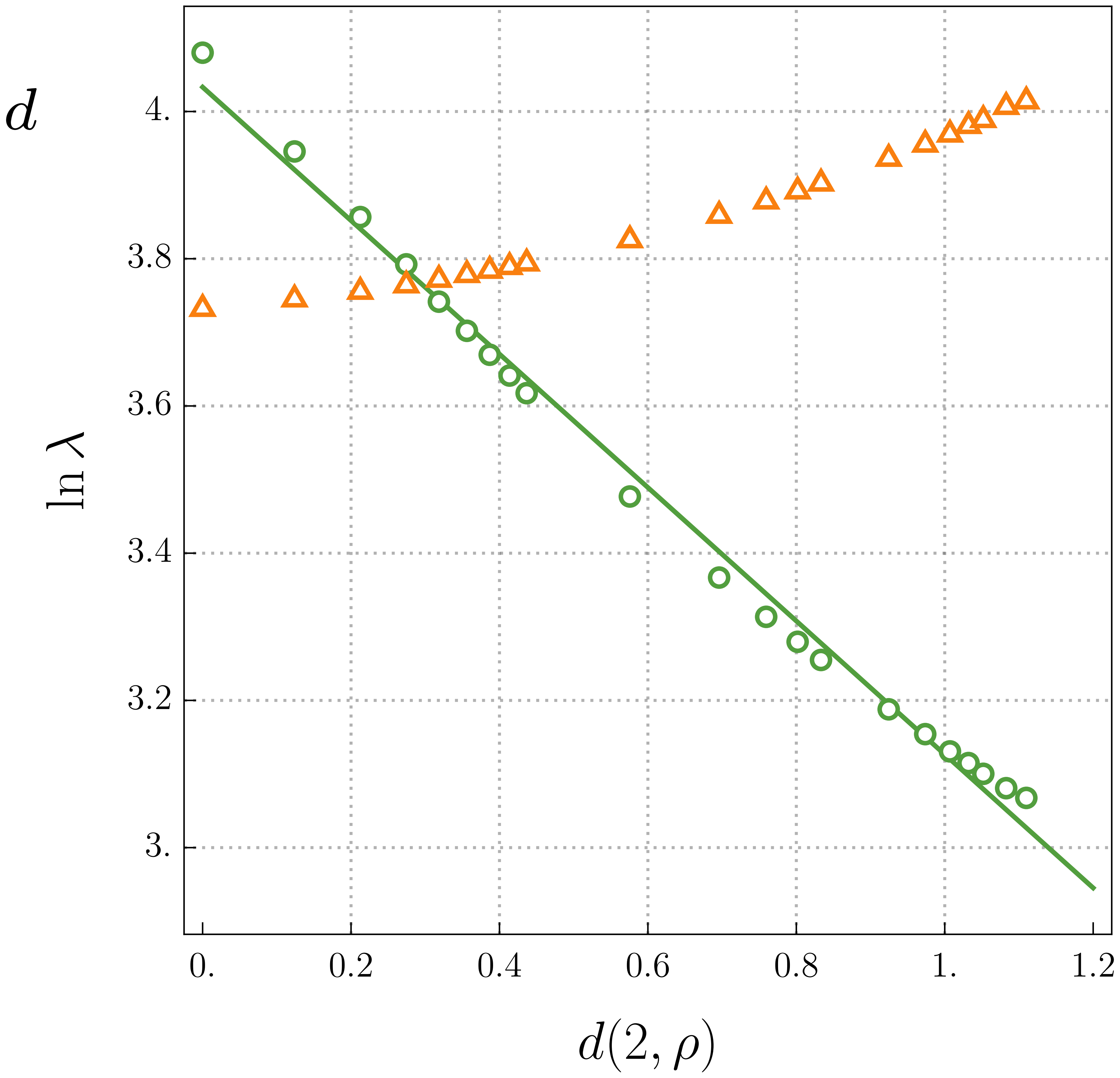
- One massless scalar (ψ)
- **Spectrum degenerate** w/ degeneracies given by **irreps of** CY **symmetry group**
- **Eigenvalues** (mass levels) **cross** in contrast (but not contradiction) to
  - No-crossing theorems in QM
  - Eigenvalue repulsion in RMT for hermitian matrices
- **States** with **large multiplicity** become **heavier**, states with **small multiplicity** become **lighter**



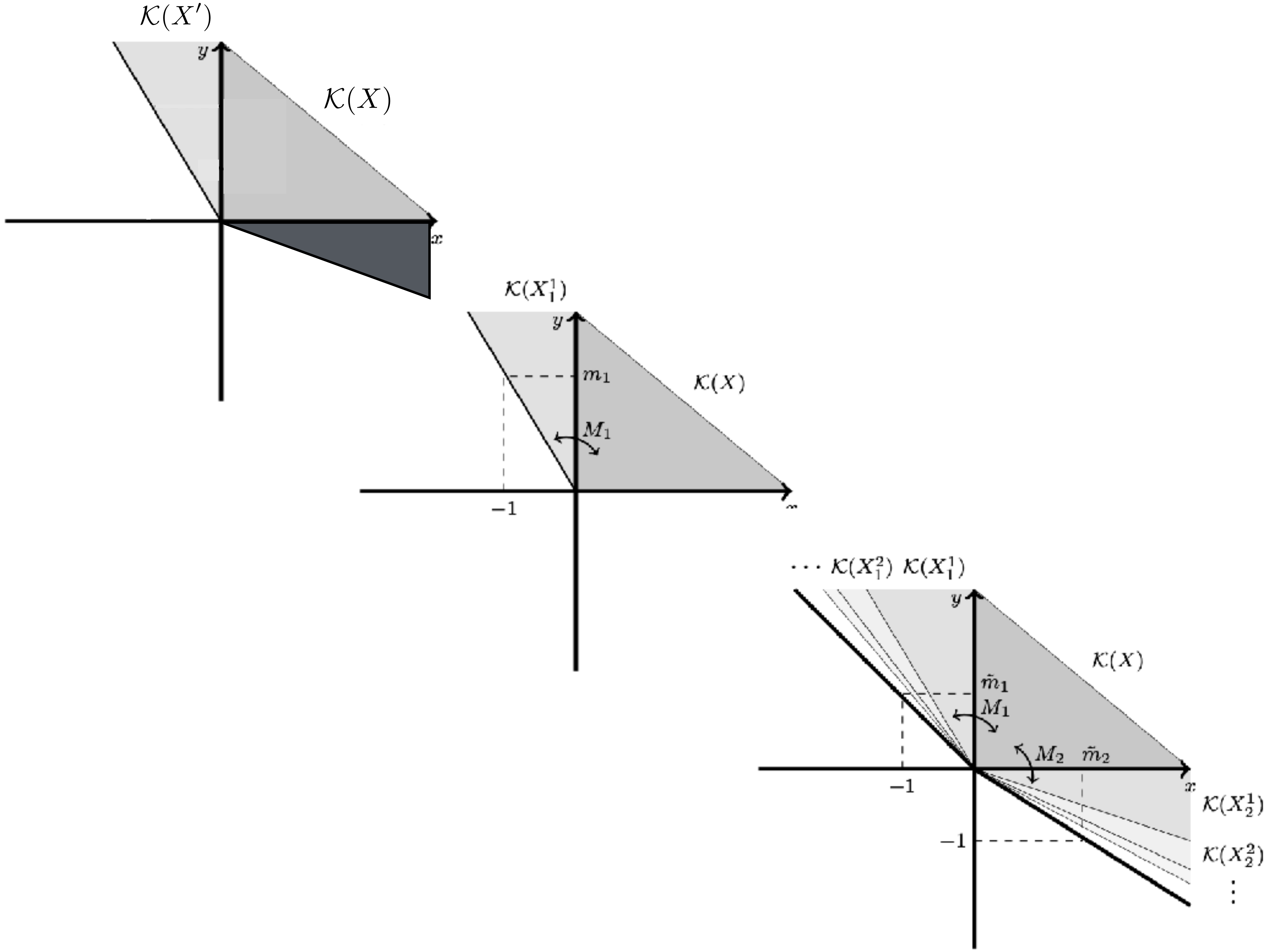
[Ashmore, FR `21]

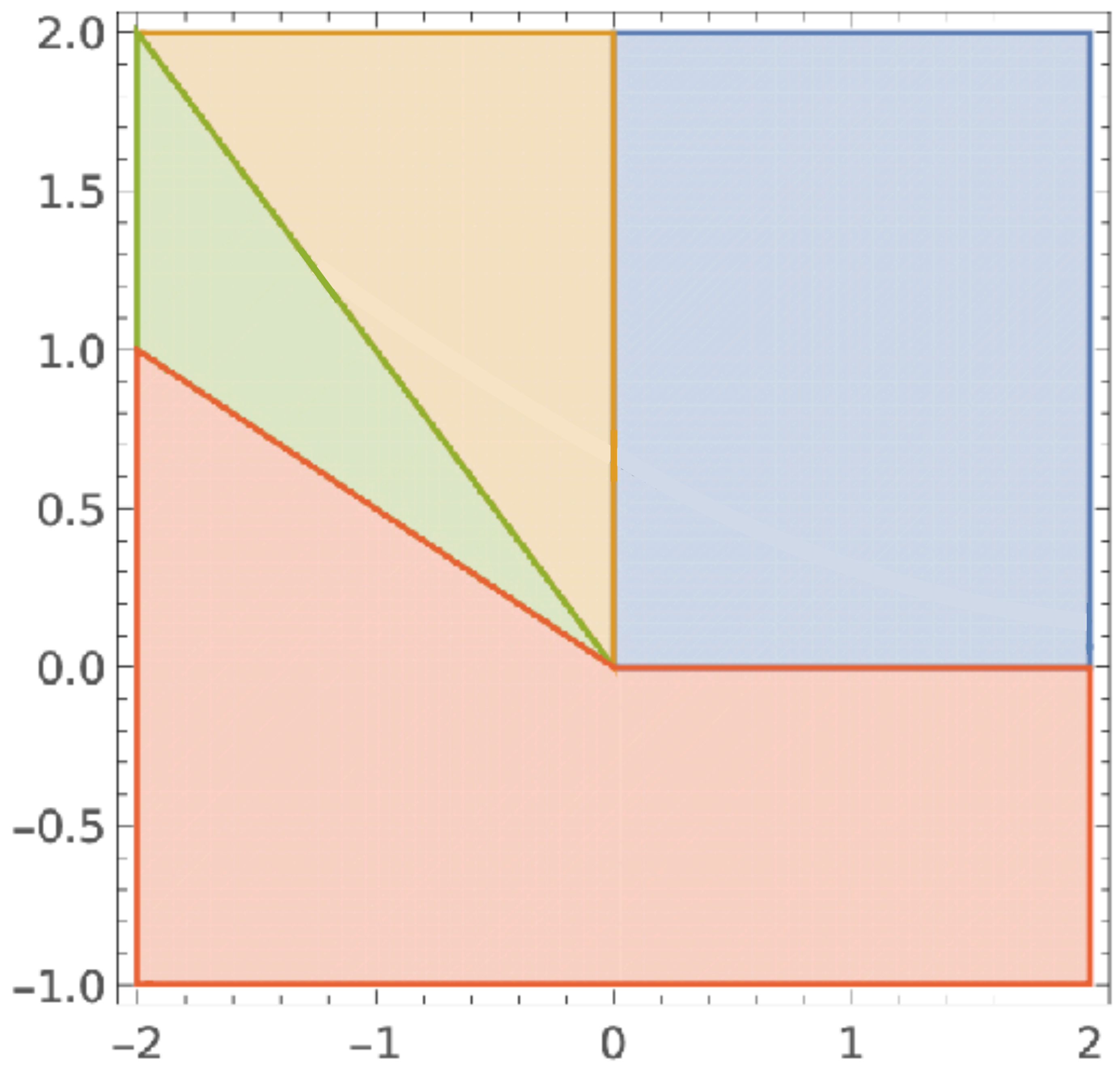
Massive KK states II

- **Scalars** become **exponentially light**
- Best fit gives $m = \lambda^{1/2} = 7.5e^{-(0.45\pm0.02)d}$
- First numerical check of the SDC
- Value very close to conjectured behavior $\alpha = 1/\sqrt{6}$ [see Muldrow's talk]
  [Andriot,Cribiori,Erkinger `20; Etheredge,Heidenreich,Kayab,Qiua,Rudelius `22]
- Interesting effect at sub-Planckian distances: mass of lightest state does not change for .3 M_P



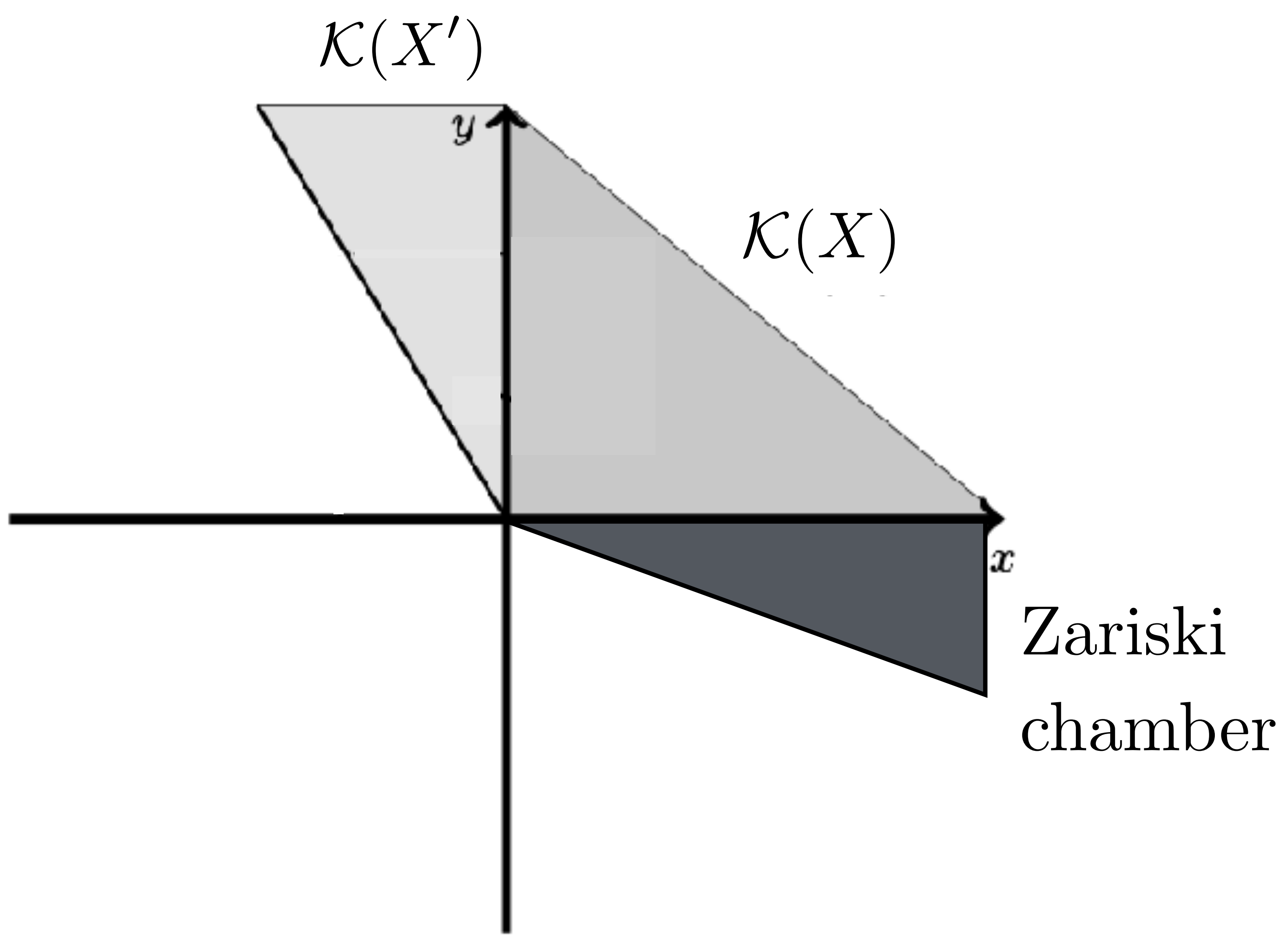
[Ashmore, FR `21]





Flops and Kähler Cones

CYs related by flops



Examples 1:

- Can flop to a non-isomorphic CY on one side
- KC ends in Zariski chamber on other side
- KC ends beyond X'

[Brodie, Constantin, Lukas, FR: 2104.03325]

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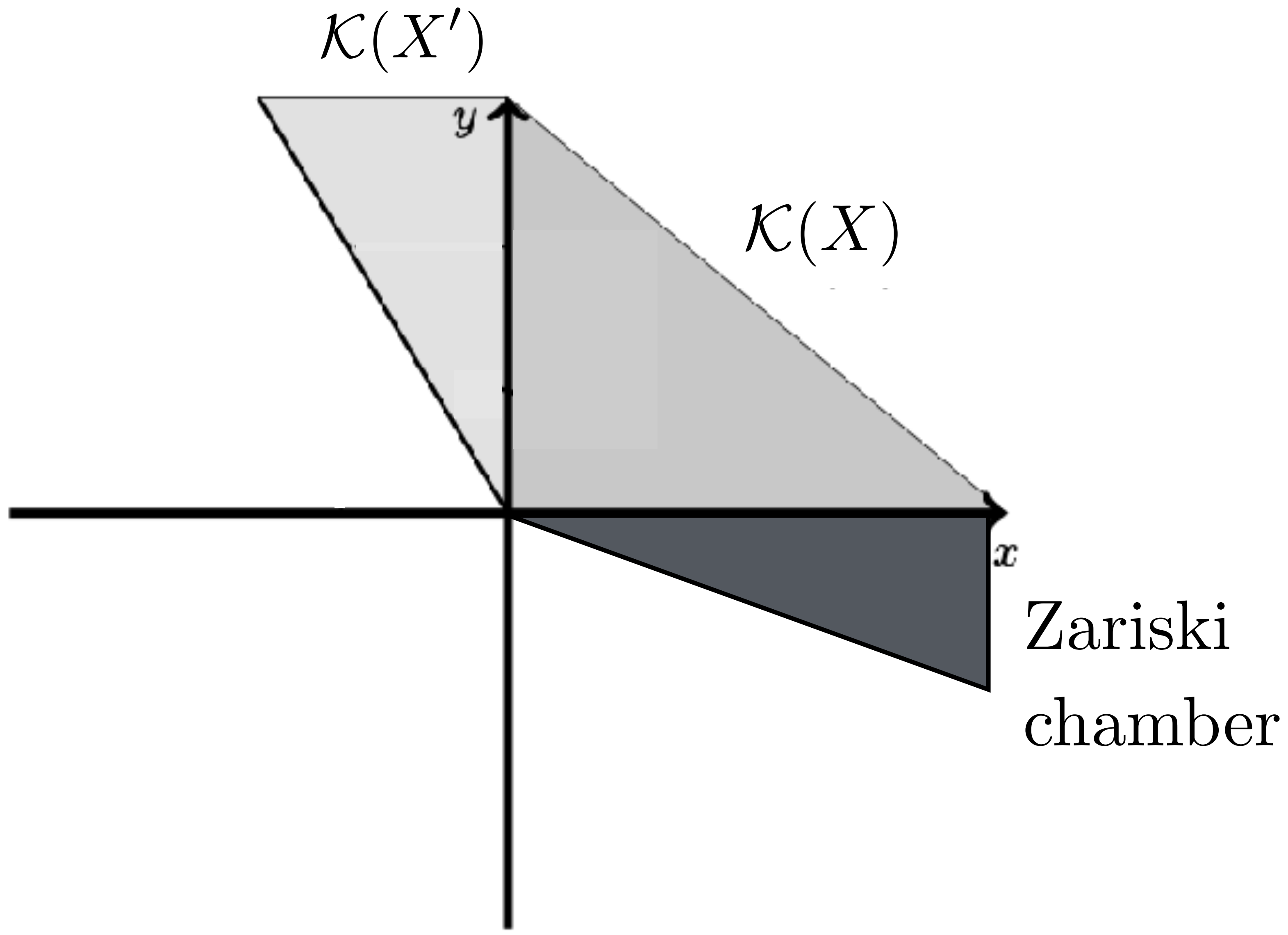

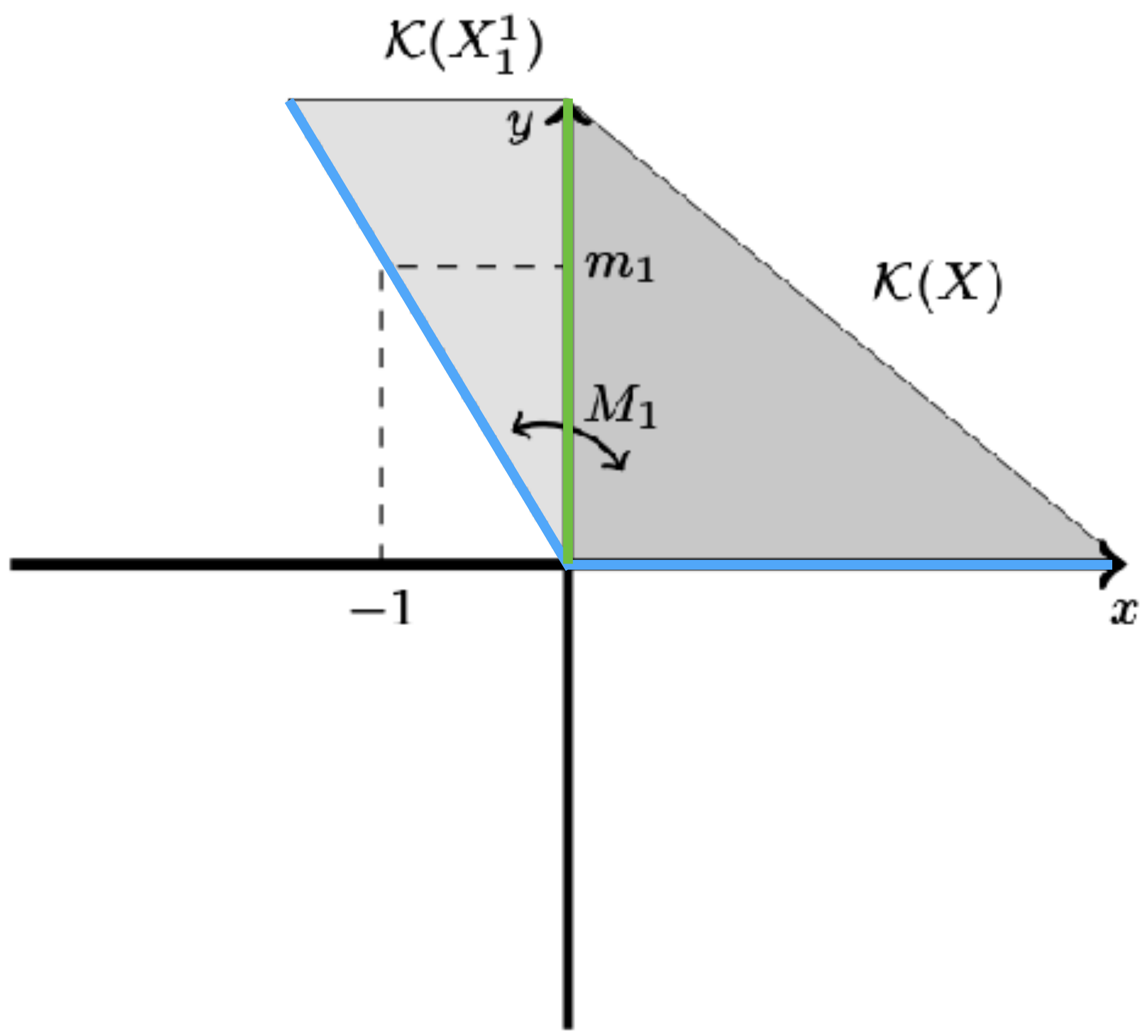

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Example 2:

- Can flop (“reflect”) to an isomorphic CY on one side
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- KC ends beyond X_1^1

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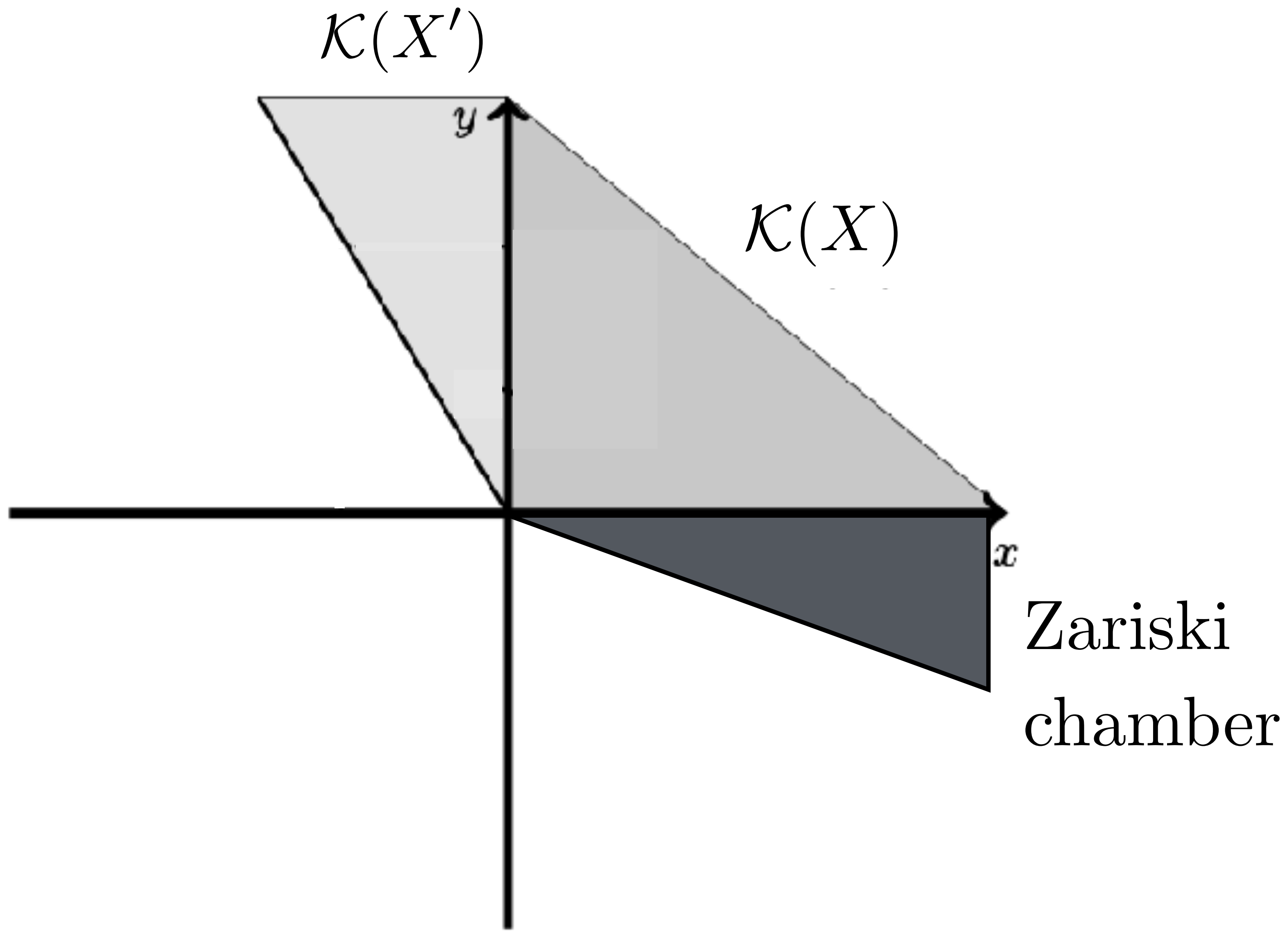

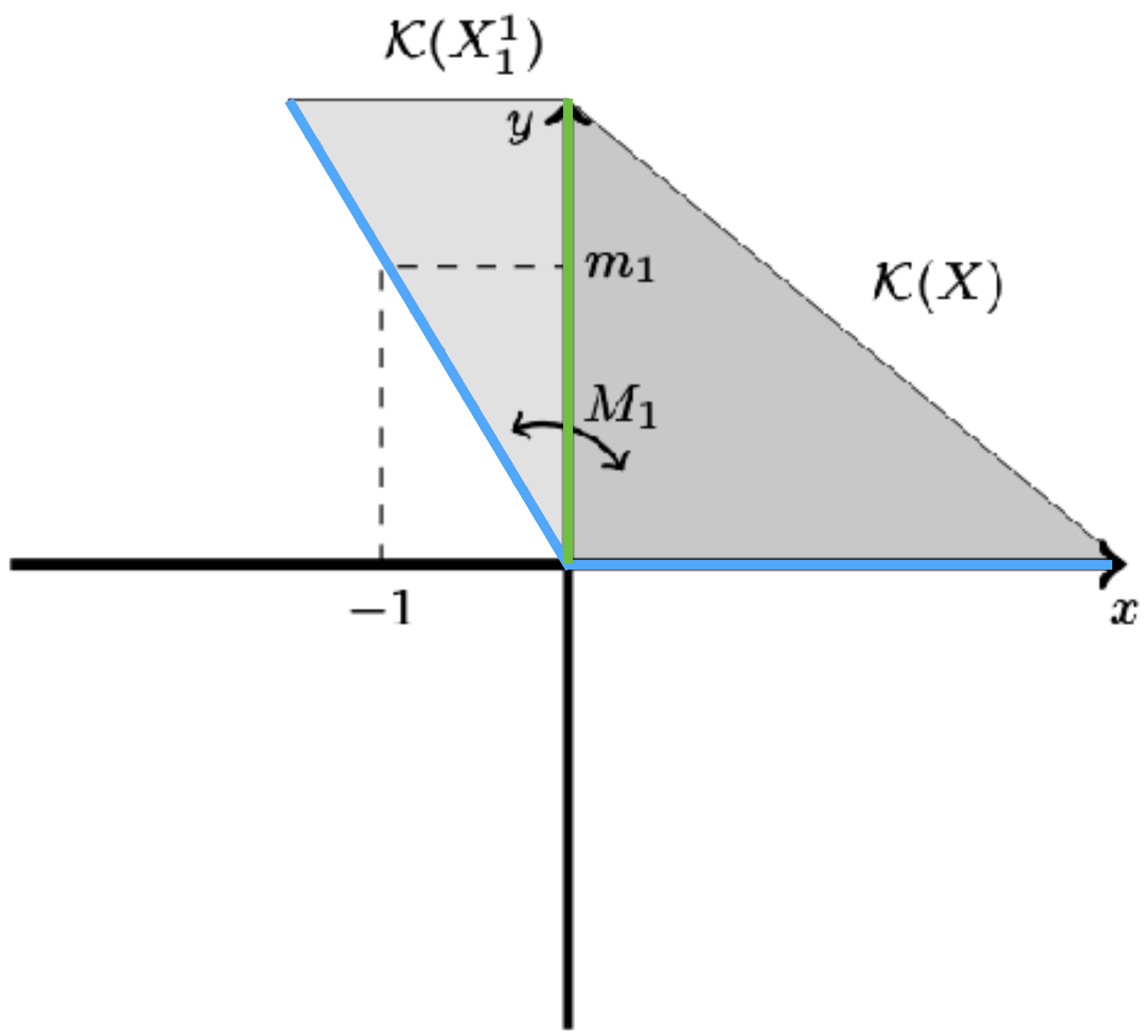

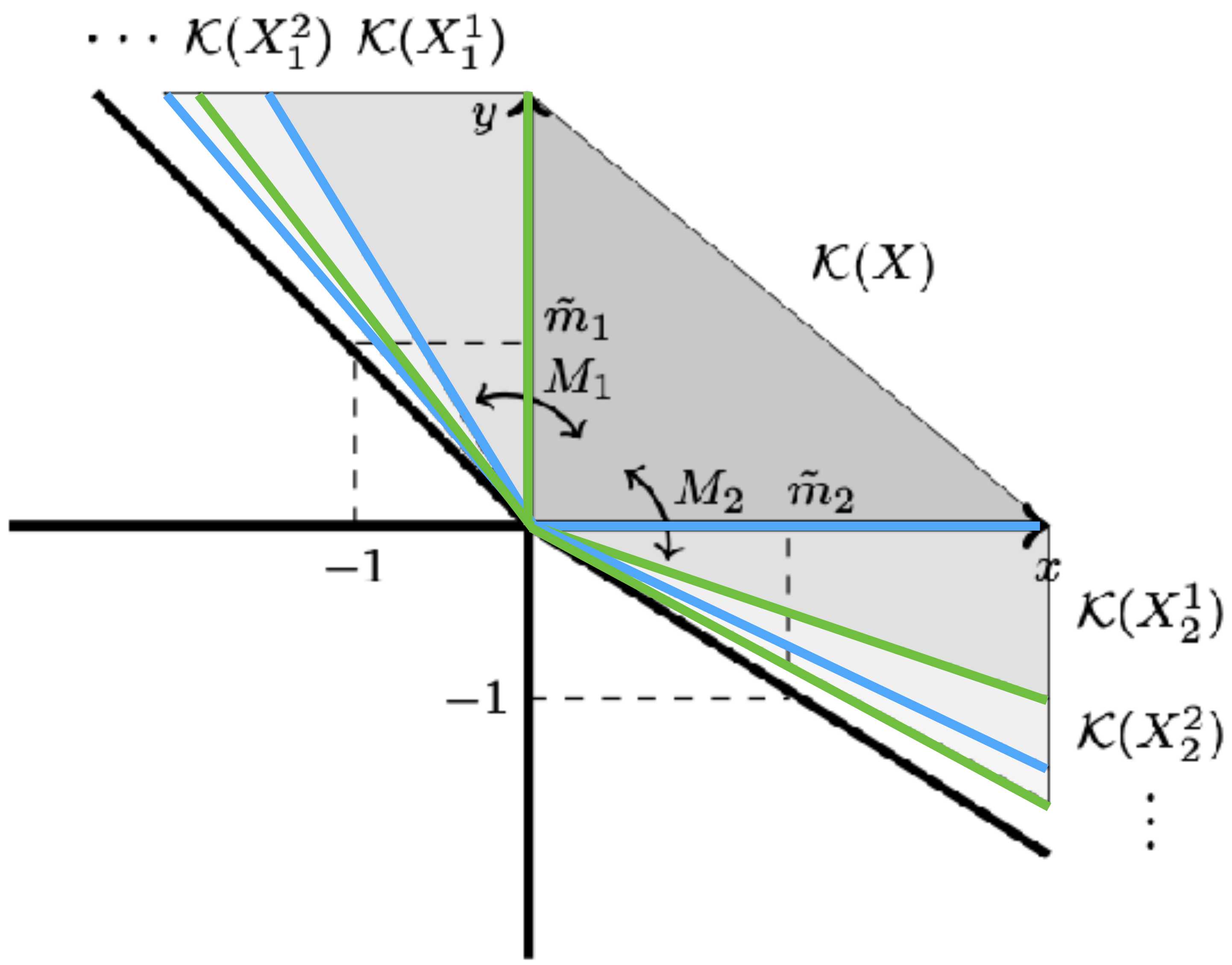

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Example 3:

- Can flop (“reflect”) to an isomorphic CY on both sides
- Repeating this gives infinitely many flops
- KC ends after infinitely many CYs

[Brodie, Constantin, Lukas, FR: 2104.03325]

Finding the reflections

- The matrices *M* that map between two isomorphic CYs *X* and *X'* can be constructed explicitly

$$\left[\begin{array}{c|ccccccc} \mathbb{P}^n & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ \vec{\mathbb{P}} & \vec{q}_1 & \vec{q}_2 & \dots & \vec{q}_{n+1} & \vec{q}_{n+2} & \dots & \vec{q}_K \end{array}\right]$$

$$M = \begin{pmatrix} -1 & \vec{0}^T \\ \sum_{k=1}^{n+1} \vec{q}_k & \mathbb{1} \end{pmatrix}$$

$$\left[\begin{array}{c|ccccccc} \mathbb{P}^n & 2 & 1 & \dots & 1 & 0 & \dots & 0 \\ \vec{\mathbb{P}} & \vec{p}_1 & \vec{p}_2 & \dots & \vec{p}_n & \vec{p}_{n+1} & \dots & \vec{p}_K \end{array}\right]$$

$$M = \begin{pmatrix} -1 & \vec{0}^T \\ 2\vec{q}_1 + \sum_{k=2}^{n} \vec{q}_k & \mathbb{1} \end{pmatrix}$$

- They depend on the Gopakumar-Vafa invariants and the intersection numbers, e.g.

$$M = \begin{pmatrix} -1 & 0 \\ m & 1 \end{pmatrix} \qquad m = \frac{2d_{122}}{d_{222}} \qquad n_{[C]} + 8n_{[2C]} = 2d_{111} - 3md_{112} + m^2 d_{122}$$

[Brodie, Constantin, Lukas, FR: 2112.12106]

Some Group Properties

- For ease of exposition, let us focus on isomorphic flops along two KC boundaries
- The matrices M_1 and M_2 generate a group with presentation

$$G = \langle M_1\,,\; M_2 \mid M_1^2 = M_2^2 = \mathbb{1} \rangle \sim \mathbb{Z}_2 \star \mathbb{Z}_2 \sim \mathbb{Z}_2 \rtimes \mathbb{Z}$$

- Let us define $S = M_1\,, \qquad T = M_1\, M_2$
- Then any word can be written as $w = T^n S^m\,, \qquad m \in \mathbb{Z}_2\,,\; n \in \mathbb{Z}$
- In particular: $T^{-n} = (T^n)^{-1}\,, \quad ST^{-n} = T^n S$
- Boundary slopes:

$$\beta_1 = \lim_{n\to\infty} \frac{c(n)}{a(n)} = \lim_{n\to\infty} \frac{d(n)}{b(n)} = -\frac{2}{m_2 + \sqrt{\frac{m_2}{m_1}(m_1 m_2 - 4)}} \qquad \beta_2 = -\frac{2}{m_1 + \sqrt{\frac{m_1}{m_2}(m_1 m_2 - 4)}}$$

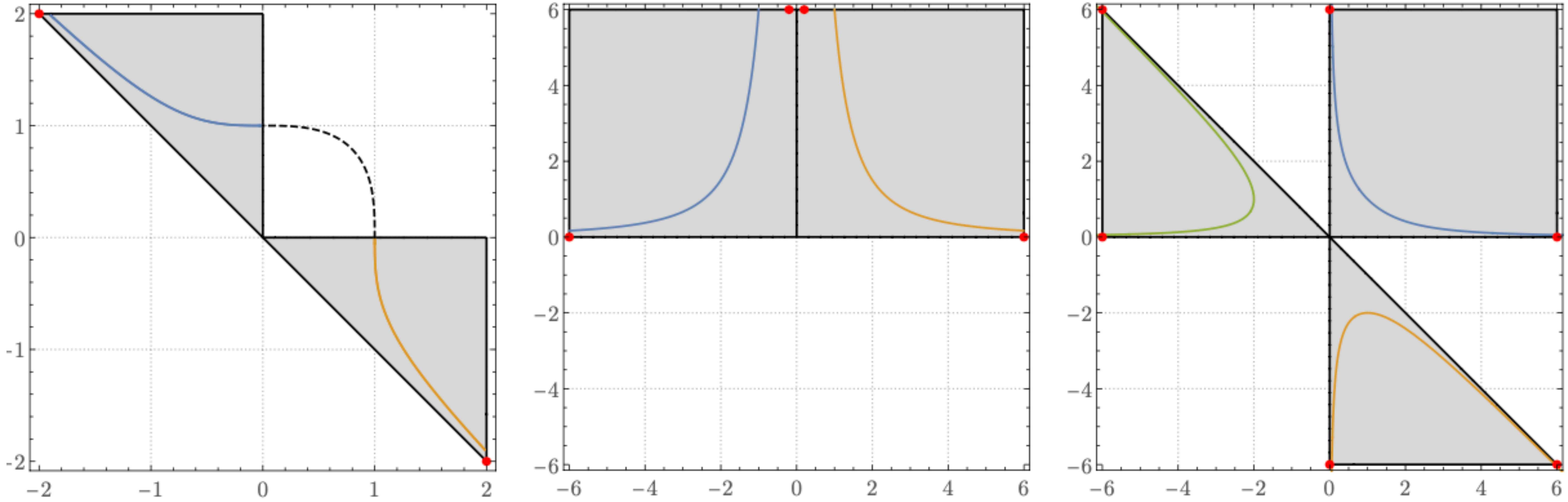
Geodesics of Picard rank 2

case	normal form $\hat{\kappa}$	(cl(κ),rk(φ))	metric $\hat{G}$	$\mathcal{K}(X)$ contained in
0	x^3	(1,1)	$\frac{1}{x^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\{\}$
1	x^3+y^3	(1,2)	$\frac{1}{\kappa^2}\begin{pmatrix} x^4-2xy^3 & 3x^2y^2 \\ 3x^2y^2 & y^4-2x^3y \end{pmatrix}$	$\{x<0, x+y>0\} \cup \{y<0, x+y>0\}$
2	x^2y	(2,2)	$\begin{pmatrix} \frac{2}{3x^2} & 0 \\ 0 & \frac{1}{3y^2} \end{pmatrix}$	$\{x \neq 0, y>0\}$
3	x^2y+xy^2	(3,2)	$\frac{1}{3}\begin{pmatrix} \frac{1}{x^2}+\frac{1}{(x+y)^2} & \frac{1}{(x+y)^2} \\ \frac{1}{(x+y)^2} & \frac{1}{y^2}+\frac{1}{(x+y)^2} \end{pmatrix}$	$\{x>0, y>0\} \cup \{x>0, x+y<0\} \cup \{y>0, x+y<0\}$

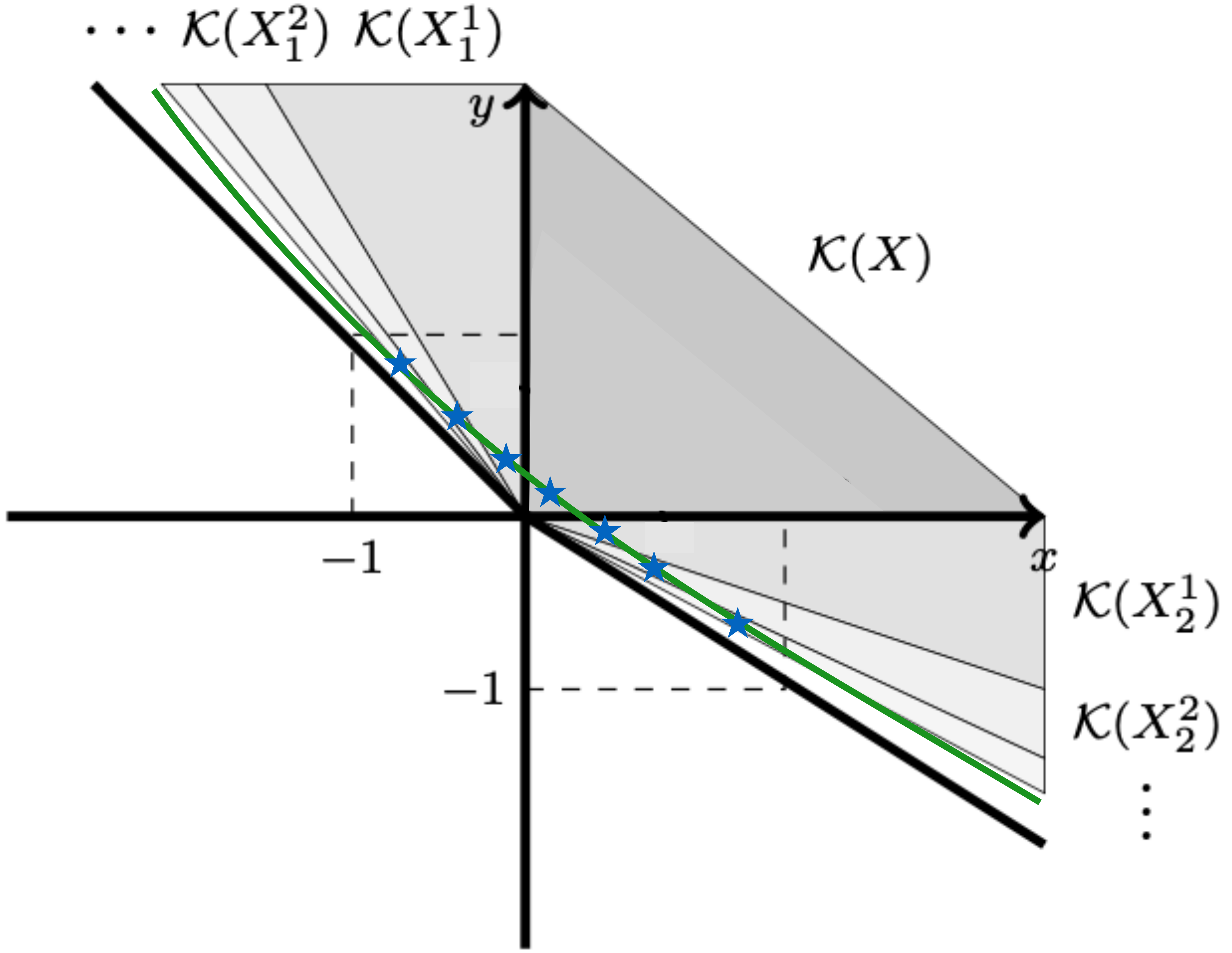
$$x = -\sqrt[3]{\sinh^2(r_k(s))}\,, \qquad y = \sqrt[3]{\cosh^2(r_k(s))}\,, \qquad r_k(s) = \sqrt{\frac{3}{8}}\,\varepsilon s + k$$

$$x = ke^{\varepsilon s}\,, \quad y = \frac{6}{k^2}e^{-2\varepsilon s}$$

$$x = \sqrt[3]{-2+6\cosh(3\varepsilon s)\ \sin\left(\frac{1}{3}\arccos\left(\tanh(-3\varepsilon s)\right)\right)}\,, \quad y_{\pm} = \frac{-x^2 \pm \sqrt{8x+x^4}}{2x}$$



Swampland Conjectures & Infinite Flops



- All CYs at ★ are the same
- The geodesic distance between two ★ is $\gtrsim \mathcal{O}(1)$
- By the Swampland Distance Conjecture, at each ★, a tower has come down by 1 e-fold, but the CYs are the same!
  [Ooguri, Vafa `06]
- Moreover, what is the fate of the symmetry group G ? Is it global?
  [Misner, Wheeler `57; Banks, Seiberg `10]

G is gauged (remnant of 11D diffeomorphisms) → divide out. Then the theories at ★ are all identified, there is a single KC, and the shortest geodesic is 0.
[Brodie, Constantin, Lukas, FR: 2104.03325]

Kawamata Morrison Cone conjecture

- What if we had flop chains involving infinitely many non-isomorphic CYs?
  $\Rightarrow$ No gauged symmetry to save the day
- The KM cone conjecture states that the extended KC only consists of finitely many isomorphism classes, so this cannot occur if the KM is true

  [Morrison `94, Kawamata `97]

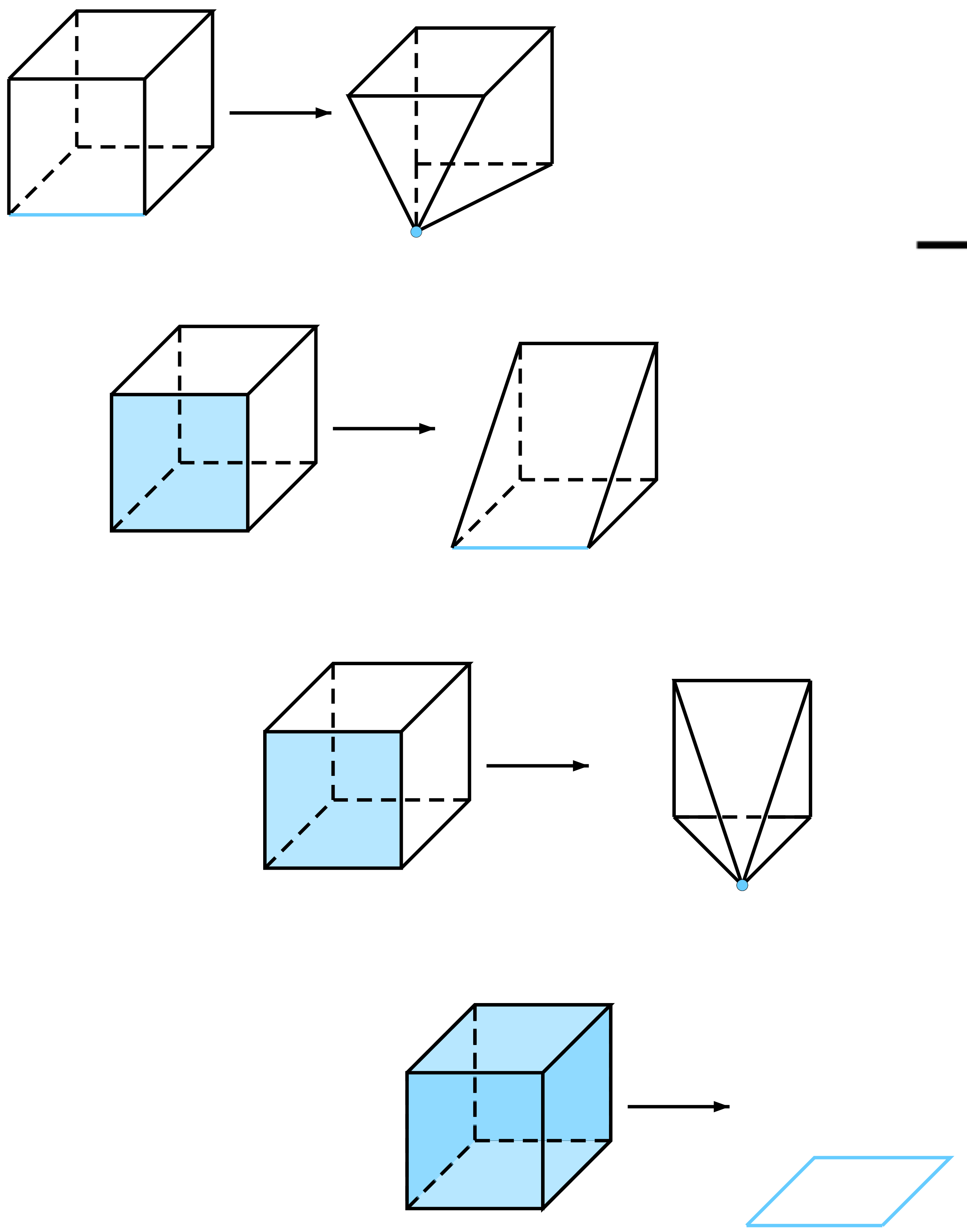
- Conversely, if there are infinitely many KCs (and the geodesic distance it takes to traverse each does not go to 0 rapidly), you would end up with a string theory somewhere in the bulk of moduli space with an arbitrarily light tower of states, which seems wrong

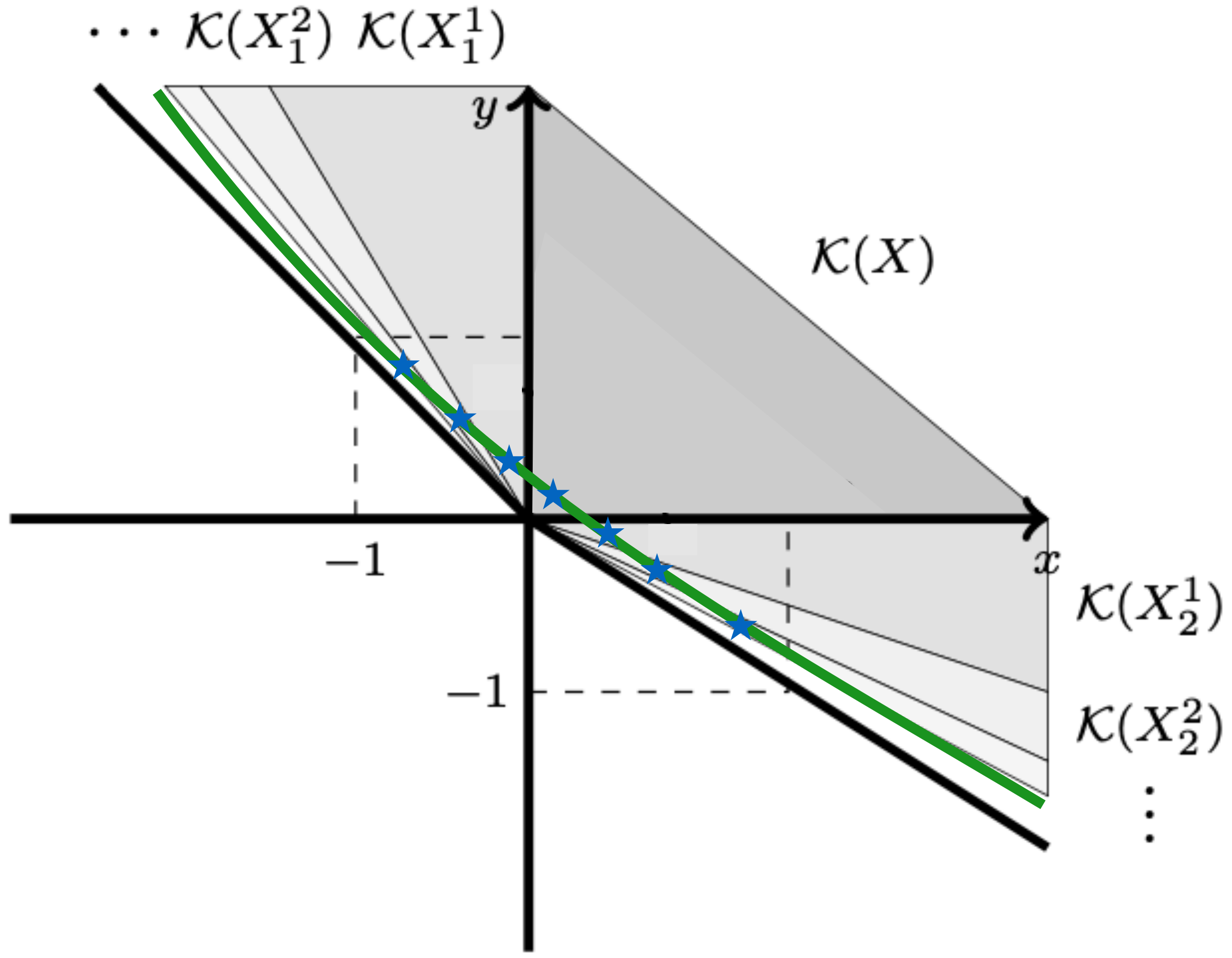
Mirror duals and modularity

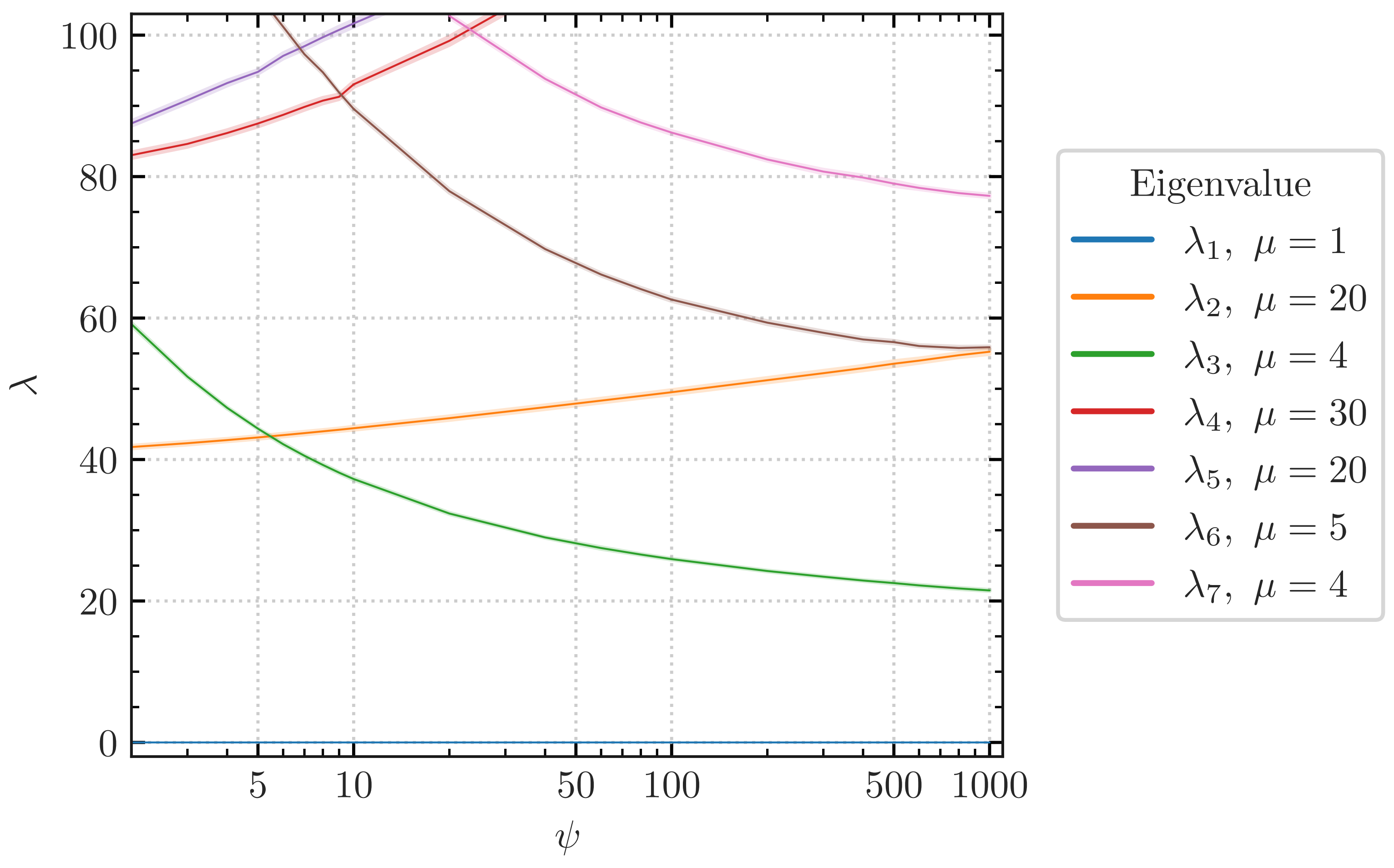
- Look at the instant contributions to the pre-potential

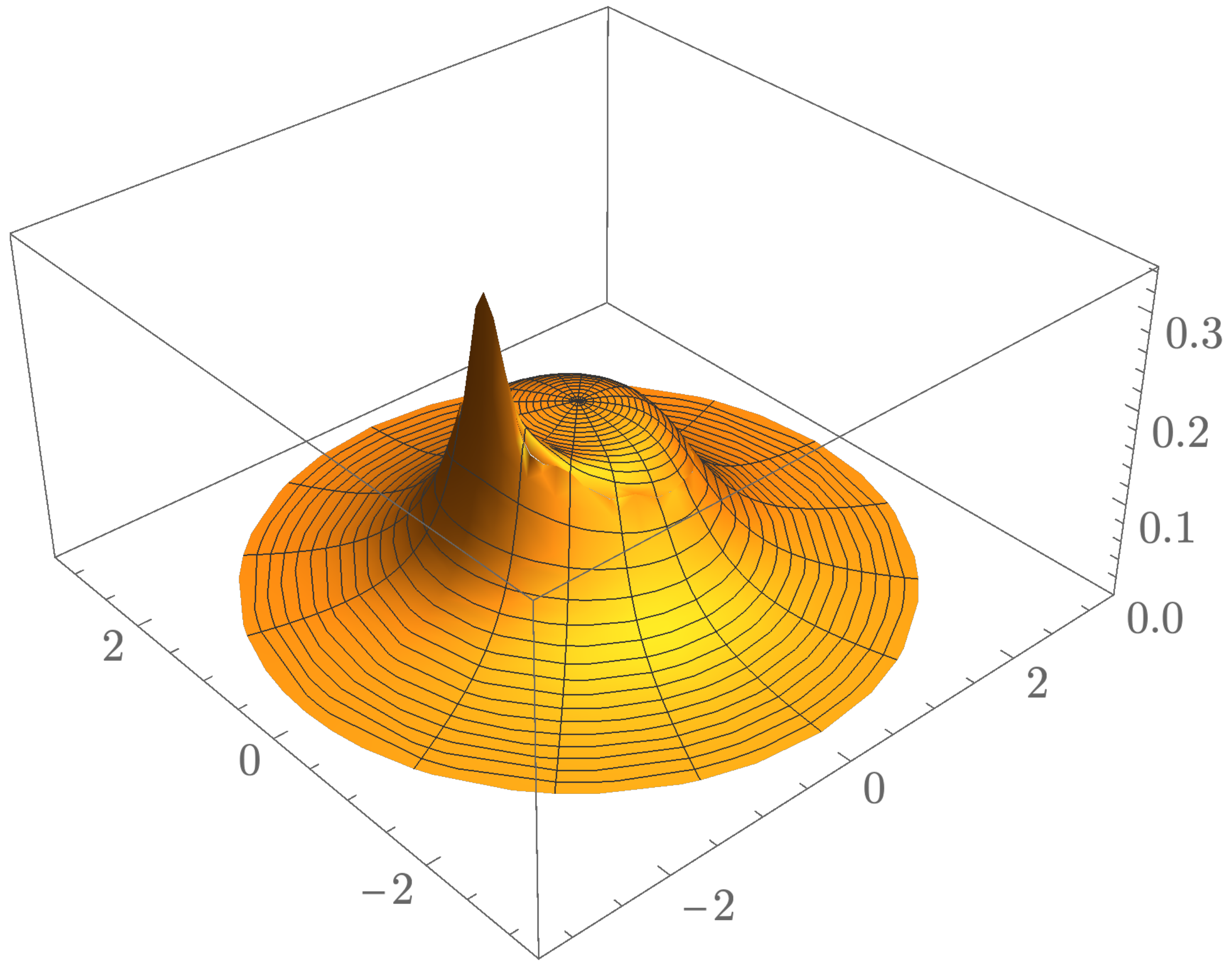
$$\mathcal{F} = \mathcal{F}_{\text{class}} + \mathcal{F}_{\text{inst}} = \mathcal{F}_{\text{class}} + \frac{1}{(2\pi i)^3} \sum_{(d_1,\dots,d_r)\in\mathbb{N}^r} n_{(d_1,\dots,d_r)} \text{Li}_3\left(\exp\left[2\pi i \sum_{j=1}^{r} d_j t_j\right]\right)$$

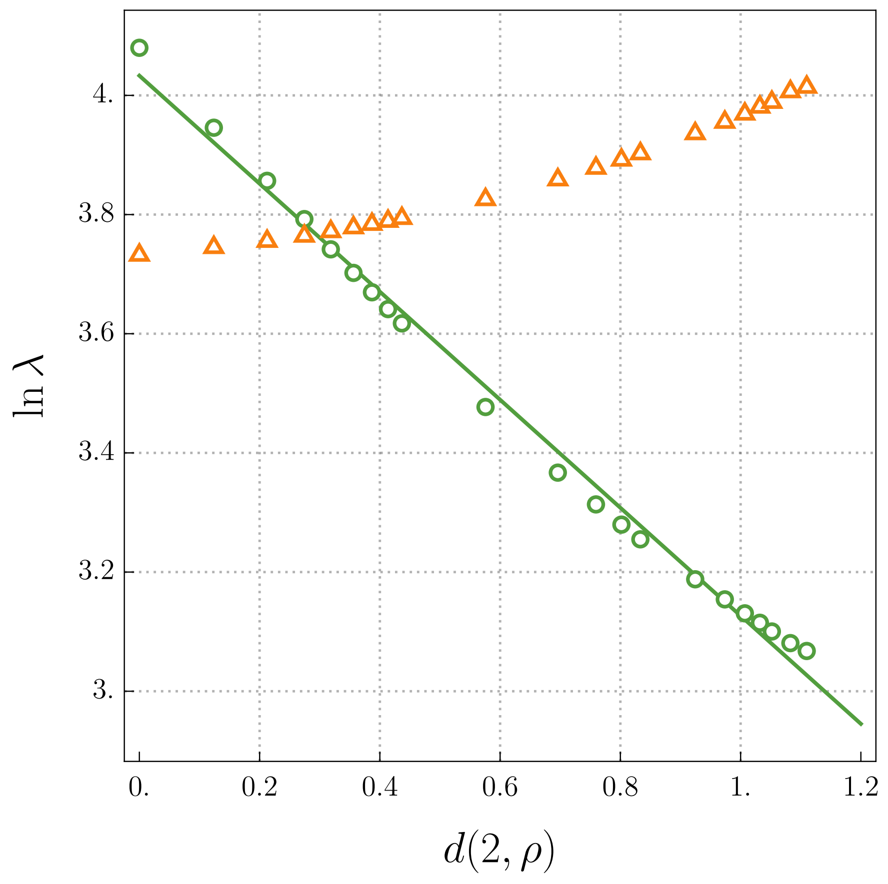
- If two curve classes are mapped onto one another, their GV invariants agree
- Can split the sum over d_i into group orbits of G
- For specific choices ($m_1 = m_2 = 2$), the exponentials in the polylog resum into theta functions ⇒ modular prepotential [work in progress]
- Interestingly, the mirror dual of the HV manifold is precisely of this type. There, its middle cohomology splits in a specific way with deep number-theoretic implications (rank 2 attractor points). [Candelas, de la Ossa, Elmi, van Straten `19; Candelas, Kuusela, McGovern `21]
- Theta-functions also arise in the 4D superpotential from ED3 branes [Naomi's talk] [Gendler, Kim, McAllister, Moritz, Stillman `22]











Conclusions

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- Constructed manifolds with infinitely many flops and their symmetry group. Related to GV invariants and intersection numbers
- Classified and solved geodesics equations for all Picard rank 2 manifolds
- The KM conjecture prevents a potential inconsistency with the SDC, and the SDC (under some assumptions) implies the KM conjecture
- Mirror-duals inherit interesting structure (LMHS, modularity, …)

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Thank you!